

A DECOMPOSITION OF THE SHARD INTERSECTION ORDER ON THE SYMMETRIC GROUP

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ABSTRACT. Inspired by work of Simion and Ullman on the lattice of noncrossing partitions, we show that the shard intersection order on the symmetric group admits a *symmetric boolean decomposition*, i.e., a partition into disjoint boolean algebras whose middle ranks coincide with the middle rank of the poset. Our decomposition also yields a new symmetric boolean decomposition of the noncrossing partition lattice.

1. INTRODUCTION

The lattice of noncrossing partitions, denoted $NC(n)$, has been an object of great interest in combinatorics and, more recently, in algebra and geometry. In [17], Simion and Ullman describe a symmetric chain decomposition of $NC(n)$. In fact, their construction might better be called a *symmetric boolean decomposition*, for in order to obtain their symmetric chain decomposition of the lattice, they first partition $NC(n)$ into disjoint boolean algebras whose middle ranks coincide with the middle rank of the lattice.

The purpose of the present note is to show that the *shard intersection order* for the symmetric group S_n , which contains $NC(n)$ as a sublattice, has a symmetric boolean decomposition. Our decomposition restricts to a decomposition of $NC(n)$, though it is different from Simion and Ullman's.

In Section 2 we give a brief overview of Reading's work [14] on the shard intersection order (W, \leq) of a Coxeter group W , which contains the lattice of W -noncrossing partitions, $NC(W)$. In Section 3, we follow Bancroft [2] in describing the shard intersection order for the case $W = S_n$. We show in Section 3.1 one way to view the classical lattice of noncrossing partitions as a sublattice of the shard intersection order (S_n, \leq) . We present the formal definition of a symmetric boolean decomposition and its immediate consequences in Section 4. That section also contains a precise statement of our main result (Theorem 2), the

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proof of which is given in Section 5. We conclude in Section 6 with remarks about generalizing the results of this paper to all finite Coxeter groups.

2. SHARDS

Introduced by Reading [14], *shards* are certain pieces of a simplicial hyperplane arrangement \mathcal{H} . For such an arrangement, the usual approach is to partition \mathcal{H} into *faces*, i.e., disjoint open subsets of the hyperplanes. Without going into detail, we form shards by again splitting up the hyperplane arrangement into subsets, each entirely contained in some hyperplane. However, these subsets are closed and they are not always disjoint. The set of intersections of shards is in bijection with the set \mathcal{R} of *regions* formed by the complement of \mathcal{H} . The lattice of intersections of shards, with partial order given by reverse containment of subspaces, thus passes to a lattice structure (\mathcal{R}, \leq) on regions.

In [14], Reading proves many general properties of this lattice, including the fact that it is graded, atomic, and coatomic. He also gives a characterization of lower intervals, computes the Möbius number, and shows that the faces of the order complex of (\mathcal{R}, \leq) are in bijection with the “pulling triangulation” of a zonotope dual to \mathcal{H} .

In the case where \mathcal{H} is the Coxeter arrangement of a root system with Coxeter group W , the regions correspond to elements $w \in W$, and so the shard intersection order gives a new lattice structure, (W, \leq) , to the Coxeter group. The shards themselves are in bijection with elements of W having exactly one descent and the rank generating function is given by the W -Eulerian polynomial,

$$W(t) = \sum_{w \in W} t^{d(w)},$$

where $d(w)$ denotes the number of descents of w . Further, the W -noncrossing partition lattice, $NC(W)$, is a sublattice of the shard intersection lattice.

Bancroft [2] has studied the lattice of shard intersections for root systems of type A_{n-1} , i.e., when the associated Coxeter group is the symmetric group. Her work gives an explicit combinatorial description to shard intersections in terms of so-called “permutation pre-orders” that helps to understand the lattice structure on S_n . Bancroft uses this model to give an EL-labeling for (S_n, \leq) .

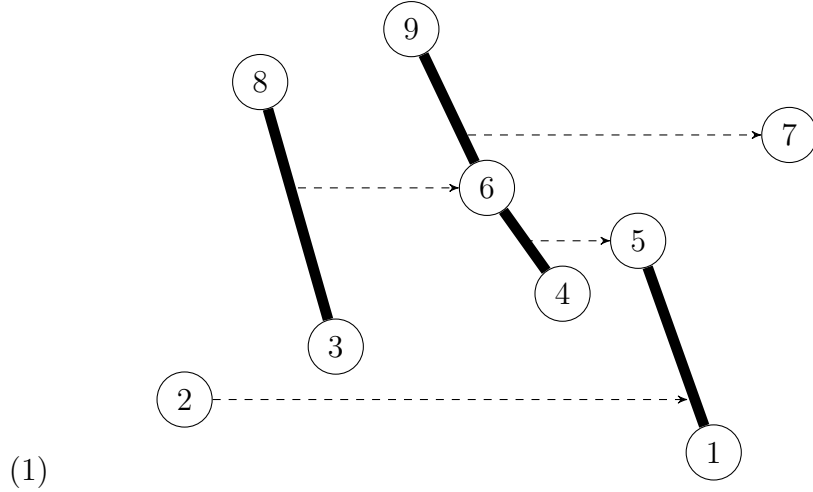
We now turn our attention to this special case.

3. THE POSET OF PERMUTATION PRE-ORDERS

We write permutations in one line notation, i.e., $w = w_1 \cdots w_n \in S_n$. A *descent* of a permutation is a letter w_i such that $w_i > w_{i+1}$ and an *ascent* is a letter w_i such that $w_i < w_{i+1}$. A (maximal) *decreasing run* of a permutation is a word found between consecutive ascents. We can highlight the decreasing runs by inserting vertical bars in ascent positions. For example, if $w = 283964517$, we write $w = 2|83|964|51|7$.

The *permutation pre-order* corresponding to w is the partial order on the decreasing runs of w formed by the transitive closure of the following relations. Namely, if $1 \leq a < b < c \leq n$ and a and c are in a decreasing run, then either b is in the same decreasing run as a and c , or the block containing b is comparable to the block containing a and c .

Visually, we represent the permutation pre-order as an array with a mark in column i (from left to right), row j (from bottom to top) if $w_i = j$. We group together any decreasing runs into blocks with thick lines:



If it is possible to draw a horizontal line to connect two decreasing runs, the block on the left is less than the block on the right in the partial order.

In all that follows we will pass freely from thinking of $w \in S_n$ as a word and as a permutation pre-order.

As Bancroft shows, permutation pre-orders correspond to type A_{n-1} shard intersections [2]. These intersections are coordinatized as certain subsets of:

$$\mathbb{R}^{n-1} \cong \mathbb{R}^n / \langle (1, 1, \dots, 1) \rangle = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n x_i = 0 \right\}.$$

Specifically, if i and j are in the same block in $w \in S_n$, then we require $x_i = x_j$. Further, if $i < k < j$ and k is not in the same block as i and j , then:

- $x_k \geq x_i = x_j$ if k appears to the left of i in w , and
- $x_i = x_j \geq x_k$ if k appears to the right of i in w .

The example shown in (1) then corresponds to the set of all points satisfying

$$x_8 = x_3 \geq x_9 = x_6 = x_4 \geq x_7,$$

$$x_9 = x_6 = x_4 \geq x_5 = x_1 \leq x_2, \quad \text{and}$$

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 = 0.$$

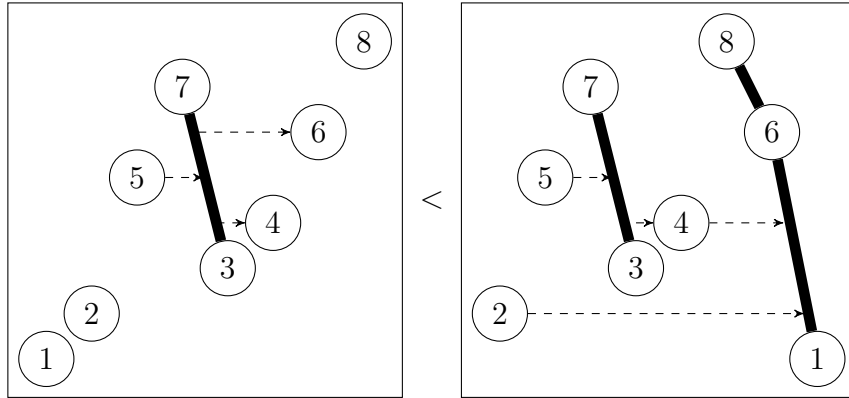
At the extremes we have $1|2|\dots|n \leftrightarrow \mathbb{R}^{n-1}$ and $n \dots 21 \leftrightarrow (0, 0, \dots, 0)$.

We can now give a partial order on S_n by reverse inclusion of the corresponding subsets of \mathbb{R}^{n-1} .

Definition 1 (Shard intersection order on permutations). *Let $u, v \in S_n$, which we think of as permutation pre-orders. Then in the shard intersection order, $u \leq v$ if:*

- (Refinement) u refines v as a set partition, and
- (Consistency) if i and j are in the same block in u , and $i < k < j$ (with k not in the same block as i and j in u), then either k is in the same block as i and j in v , or k is on the same side of i and j in v as in u .

That is, the partial order on blocks of v may be obtained by merging some blocks in the partial order on the blocks of u . For example, $1|2|5|73|4|6|8 < 2|5|73|4|861$, i.e.,



because we can obtain the pre-order on the right by merging the 8, the 6, and the 1. This new block had to be to the right of the block with the 3 and the 7 because the 6 was already to the right. The new block has to be comparable to the 2 and comparable to the 4, but it could have

appeared to the right or the left of the 2, and to the right or the left of the 4. These choices give rise to other permutation pre-orders (with the same set of blocks) that lie above $1|2|5|73|4|6|8$, namely, $5|73|4|861|2$, $2|5|73|861|4$, and $5|73|861|2|4$.

3.1. Noncrossing partitions. Reading [14] shows generally that the lattice of noncrossing partitions of type W is an induced sublattice of (W, \leq) . For $W = S_n$, this fact can be realized by restricting to the set of 231-avoiding permutations, which are well-known to be in bijection with classical noncrossing partitions.

We say $w \in S_n$ is 231-avoiding if there is no triple of indices $i < j < k$ such that $w_k < w_i < w_j$. Let $S_n(231)$ denote the set of 231-avoiding permutations. For example, $51243 \in S_5(231)$ and $31524 \notin S_5(231)$.

A *noncrossing partition* $\pi = \{R_1, R_2, \dots, R_k\}$, is a set partition of $\{1, 2, \dots, n\}$, such that if $\{a, c\} \subseteq R_i$ and $\{b, d\} \subseteq R_j$, with $1 \leq a < b < c < d \leq n$, then $i = j$. That is, two pairs of numbers from distinct blocks cannot be interleaved. Let $NC(n)$ denote the set of all noncrossing partitions of $\{1, 2, \dots, n\}$. For example, $\{\{1, 5\}, \{2\}, \{3, 4\}\} \in NC(5)$, while $\{\{1, 3\}, \{2, 5\}, \{4\}\} \notin NC(5)$.

The lattice of noncrossing partitions is the partially ordered set $(NC(n), \preceq)$ with $\sigma \preceq \tau$ if σ refines τ as a set partition.

Define a bijection $\phi : S_n(231) \rightarrow NC(n)$ by mapping the decreasing runs of a permutation to blocks in a partition. See Figure 1. Specifically, if $w = d_1|d_2|\dots|d_k$, where the d_i are the blocks of decreasing runs of w , then letting D_i denote the set of letters of d_i , we have $\phi(w) = \{D_1, D_2, \dots, D_k\}$. For example, if $w = 421|3|765|98$, then

$$\phi(w) = \{\{1, 2, 4\}, \{3\}, \{5, 6, 7\}, \{8, 9\}\}.$$

The inverse map takes the blocks of π , lists each block in decreasing order, and then orders the blocks from left to right according to the smallest element in the block. For example, if

$$\pi = \{\{1, 7, 9\}, \{2, 3\}, \{4, 6\}, \{5\}, \{8\}\},$$

then $\phi^{-1}(\pi) = 971|32|64|5|8$.

It is straightforward from the definitions that if $u \leq v$ in (S_n, \leq) , then $\phi(u) \preceq \phi(v)$ in $(NC(n), \preceq)$. It is only slightly more subtle to check that the converse is true. We have the following.

Theorem 1 ([14], Theorem 8.5). *The lattice of noncrossing partitions $(NC(n), \preceq)$ is isomorphic to the induced sublattice $(S_n(231), \leq)$ inside (S_n, \leq) .*

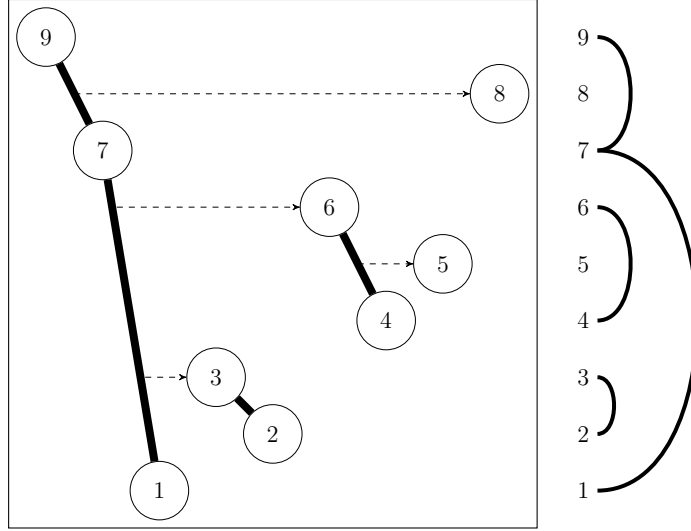


FIGURE 1. The decreasing runs of a 231-avoiding permutation form a noncrossing partition.

In Figure 2, we see the shard intersection order (S_4, \leq) , with the lattice of noncrossing partitions $(NC(4), \preceq) \cong (S_n(231), \leq)$ highlighted in bold.

4. SYMMETRIC BOOLEAN DECOMPOSITION

Suppose (P, \leq) is a graded poset of rank n , and let B_j denote the boolean algebra on j elements.

Definition 2 (Symmetric boolean decomposition). *A symmetric boolean decomposition of (P, \leq) is a partition of the elements of P , $\{P_1, \dots, P_k\}$ (i.e., $P_i \cap P_j = \emptyset$, $\cup_i P_i = P$), such that for each $i = 1, \dots, k$:*

- *there is a j , $0 \leq j \leq n/2$, such that $|P_i| = 2^{n-2j}$,*
- *there is a unique minimum element of P_i and its rank is j ,*
- *the boolean algebra B_{n-2j} is isomorphic to a subposet of the induced poset (P_i, \leq) in (P, \leq) .*

For example, in Figure 3, poset (a) has a symmetric boolean decomposition (in bold), while (b) and (c) do not. Note that poset (b) differs from (a) only in one cover relation.

It is well known that a boolean algebra admits a symmetric chain decomposition (see, e.g., [7]), hence the following observation.

Observation 1. *If (P, \leq) admits a symmetric boolean decomposition, then it admits a symmetric chain decomposition.*

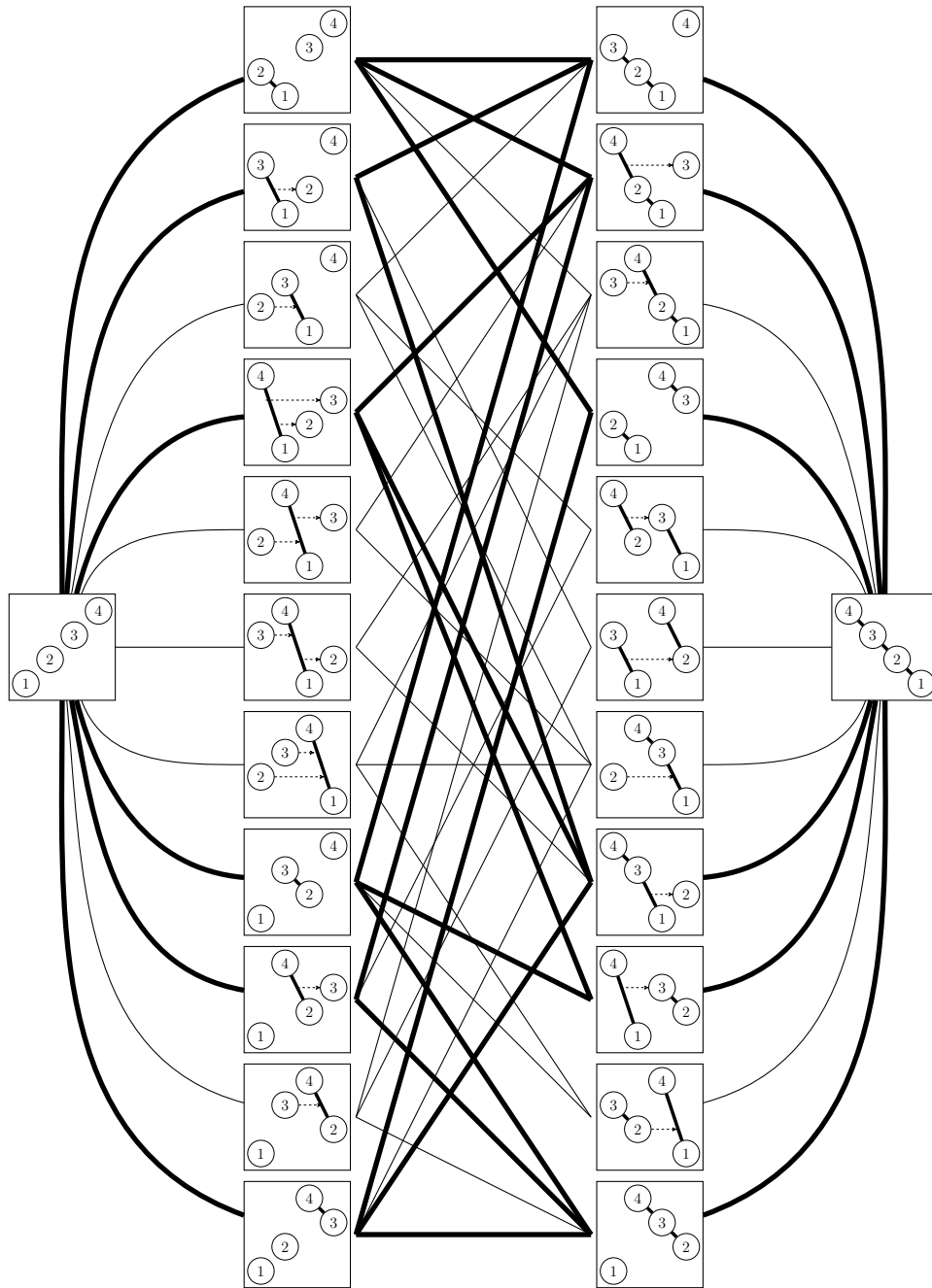


FIGURE 2. The shard intersection lattice for S_4 contains the lattice of noncrossing partitions.

Along with having a symmetric chain decomposition, there must be “enough” vertices of each rank for a given poset to admit a symmetric

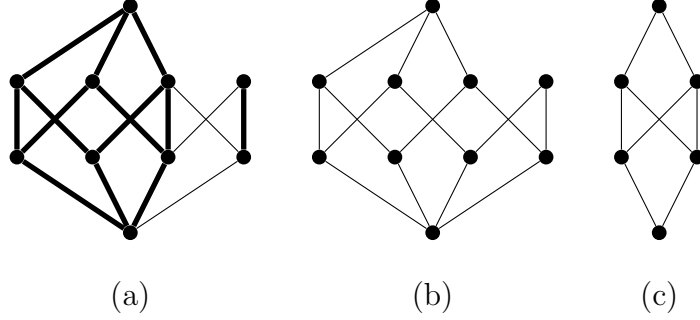


FIGURE 3. Posets with and without symmetric boolean decompositions.

boolean decomposition. The notion of γ -nonnegativity encapsulates this numeric condition.

Observation 2. *Let (P, \leq) be a graded poset of rank n . If (P, \leq) admits a symmetric boolean decomposition, then there exist nonnegative integers γ_j such that*

$$\sum_{p \in P} t^{k(p)} = \sum_{0 \leq j \leq n/2} \gamma_j t^j (1+t)^{n-2j}.$$

In fact, γ_j is the number of subsets P_i in the symmetric boolean decomposition for which $|P_i| = 2^{n-2j}$.

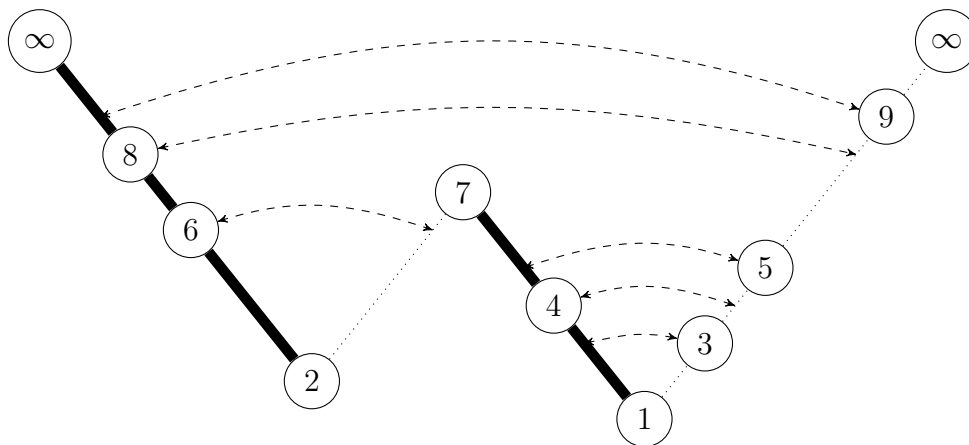
Note that while having a symmetric chain decomposition implies that the rank generating function for (P, \leq) is symmetric and unimodal, γ -nonnegativity is a stronger condition. This property has been of some interest since Gal conjectured that the h -polynomials of flag spheres are γ -nonnegative [6]. See also [3, 10, 11, 13, 18].

In the case of $NC(n)$, its rank generating function is known to be the h -polynomial of the *associahedron*. As Postnikov, Reiner, and Williams observe, γ -nonnegativity of its h -polynomial follows from Simion and Ullman's symmetric boolean decomposition of $NC(n)$. See [13, Proposition 11.14] and Sections 2 and 3 of [17].

Similarly, the rank generating function of the shard intersection order of the symmetric group S_n is the *Eulerian polynomial*

$$A_{n-1}(t) = \sum_{w \in S_n} t^{d(w)},$$

which is known to be the h -polynomial of the *permutahedron*. It is known that the Eulerian polynomials are γ -nonnegative, as first shown by Foata and Schützenberger [4, Théorème 5.6]. Their proof of this



fact used an elegant combinatorial argument that we can use to prove the following, which is the main result of this paper.

5. PROOF OF THEOREM 2

Formally, given $w = w_1 \cdots w_n \in S_n$, we say a letter w_i is a *peak* if $w_{i-1} < w_i > w_{i+1}$ and it is a *valley* if $w_{i-1} > w_i < w_{i+1}$. Otherwise we say w_i is *free*. Using the convention that $w_0 = w_{n+1} = \infty$, we see that w cannot begin or end with a peak.

$$F(w) = \{1 \leq w_i = j \leq n : w_{i-1} < w_i < w_{i+1} \text{ or } w_{i-1} > w_i > w_{i+1}\}.$$

If $w_i = j$ is free, then $H_j(w)$ denotes the permutation obtained by moving j directly across the adjacent valley(s) to the nearest mountain slope of the same height. More precisely, we have the following.

Definition 3 (Valley hopping). *Let $w \in S_n$, and let $w_i = j$ be a free letter of w . Define the operator $H_j(w)$ as follows:*

- *if $w_i = j$ lies on a downslope, $w_{i-1} > w_i > w_{i+1}$, we find the smallest $k > i$ such that $w_k < j < w_{k+1}$, and*

$$H_j(w) = w_1 \cdots w_{i-1} w_{i+1} \cdots w_k j w_{k+1} \cdots w_n,$$

- *if $w_i = j$ lies on an upslope, $w_{i-1} < w_i < w_{i+1}$, we find the largest $k < i$ such that $w_{k-1} > j > w_k$, and*

$$H_j(w) = w_1 \cdots w_{k-1} j w_k \cdots w_{i-1} w_{i+1} \cdots w_n.$$

Clearly, if $j, l \in F(w)$, $H_j^2(w) = w$ and $H_j(H_l(w)) = H_l(H_j(w))$. Thus, for any $J = \{j_1, \dots, j_k\} \subseteq F(w)$, we can define the operation $H_J(w) = H_{j_1} \cdots H_{j_k}(w)$. Also, observe that $F(H_J(w)) = F(w)$, i.e., $H_J(w)$ has the same set of free letters as w .

Define a relation $v \sim w$ if there is a sequence of hops on free letters that transforms w into v . It is easy to see that \sim is an equivalence relation. Let P_w denote the hop-equivalence class of w . If w has r peaks, it has $r + 1$ valleys, and hence $n - 1 - 2r$ free letters. Therefore we see that $|P_w| = 2^{n-1-2r}$. See Figure 5 for the class of $w = 13527846$.

For each such class, there is a unique element with the minimal number of descents, r , corresponding to having each free letter lie on an upslope. Let \widehat{S}_n denote the set of these descent-minimal representatives:

$$\widehat{S}_n = \{w \in S_n : w_1 < w_2 \text{ and if } w_{i-1} > w_i, \text{ then } w_i < w_{i+1}\}.$$

These are the permutations for which every descent is also a peak. We claim that the collection of all hop-equivalence classes of elements in \widehat{S}_n ,

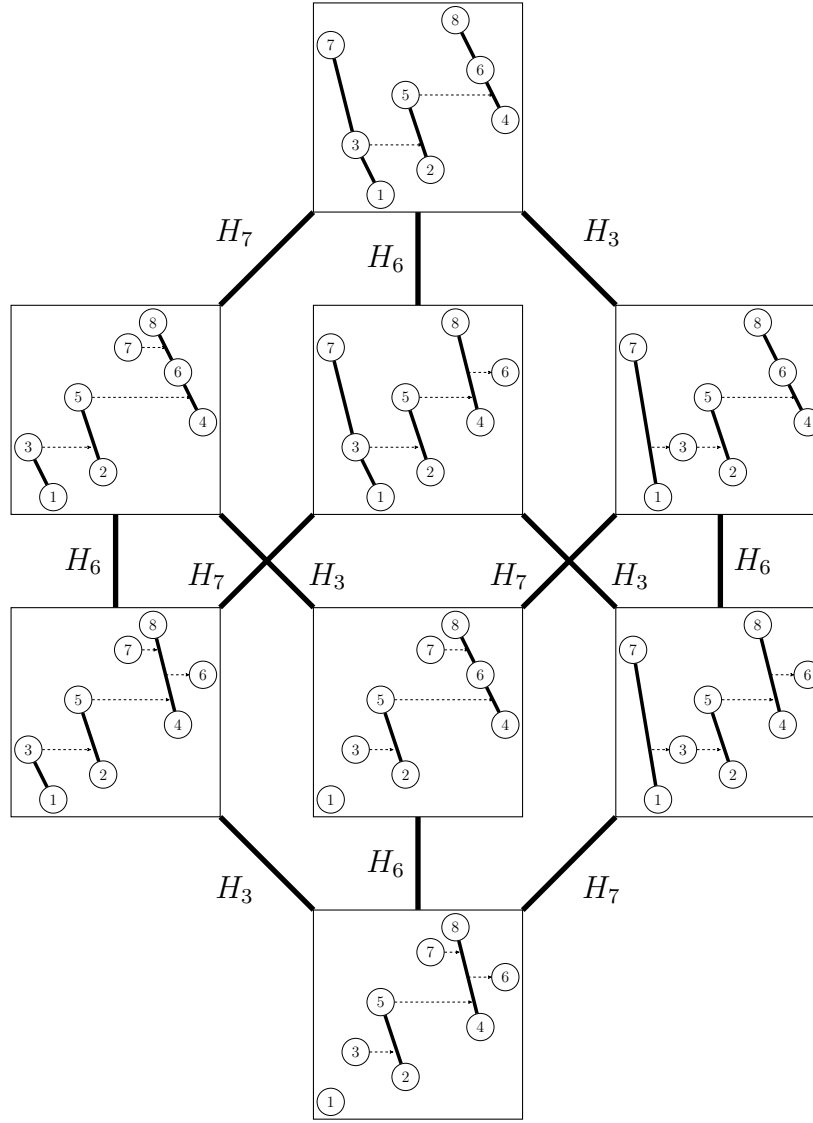
$$\mathcal{P} = \{P_w : w \in \widehat{S}_n\},$$

gives a symmetric boolean decomposition of (S_n, \leq) .

Lemma 1 (Partitions S_n). *The collection \mathcal{P} is a partition of S_n .*

Proof. Since valley-hopping defines an equivalence relation, we know the set of all hop-equivalence classes partitions S_n . All we need is to show that for every $w \in S_n$, there is a unique $u \in \widehat{S}_n$ such that $w \sim u$. The desired element is given as follows. Let $J \subseteq F(w)$ be the set of all free letters that lie on a downslope. Then $u = H_J(w) \in \widehat{S}_n$, and for any $K \neq J$, $H_K(w) \notin \widehat{S}_n$ since it has some letter on a downslope. \square

Together with Lemma 1, the following lemma establishes the first claim of Theorem 2.

FIGURE 5. The induced poset (P_w, \leq) for $w = 13527846$.

Lemma 2 (Boolean isomorphism). *Let $w \in \widehat{S}_n$ with r descents. If $J \subseteq K \subseteq F(w)$, then $H_J(w) \leq H_K(w)$ in the shard intersection order (S_n, \leq) . Conversely, if $H_J(w) \leq H_K(w)$ in (S_n, \leq) , then $J \subseteq K$. In other words, the induced poset $(P_w, \leq) \cong B_{n-1-2r}$.*

Proof. For the first claim, that $H_J(w) \leq H_K(w)$ if $J \subseteq K$, it suffices to assume $u \in P_w$ with j on an upslope and show $u < H_j(u)$ in the shard intersection order.

If $u_i = j$ is on an upslope in u , then it forms its own (singleton) decreasing run, while in $H_j(u)$, the letter j is part of a block of size at least two (including at least the nearest valley to its right). The hop operator H_j leaves the permutation pre-order of u unchanged in all other ways, so we see that u refines $H_j(u)$ and their partial orders on decreasing runs are consistent; i.e., from Definition 1 we see $u < H_j(u)$ in (S_n, \leq) .

Now suppose $J \not\subseteq K$. We will show $u = H_J(w) \not\leq H_K(w) = v$. In particular let $j \in J$, $j \notin K$. Then in u , j lies on a downslope, while in v , j lies on an upslope. This means in particular that j forms a singleton block in v , while it is part of a block of size at least two in u . Thus, as pre-orders, u cannot be a refinement of v , and Definition 1 shows $u \not\leq v$ in (S_n, \leq) . \square

We now show the second claim of Theorem 2, that our symmetric boolean decomposition restricts to noncrossing partitions. Because of Lemma 2 we need only show that

$$\mathcal{P}(231) = \{P_w : w \in \widehat{S}_n \cap S_n(231)\}$$

is a partition of $S_n(231)$, the set of 231-avoiding permutations. We already know from Lemma 1 that the classes P_w , $w \in \widehat{S}_n$, are pairwise disjoint, so it only remains to check that if $w \in S_n(231)$, then its hop-equivalence class P_w is entirely contained in $S_n(231)$. This is established with the following lemma.

Lemma 3 (Restriction to 231-avoiders). *Let $w \in S_n(231)$. Then $P_w \subseteq S_n(231)$. That is, valley hopping gives an equivalence relation on the set of 231-avoiding permutations.*

Proof. It suffices to show that if $w \notin S_n(231)$, $H_s(w) \notin S_n(231)$ for all $s \in F(w)$.

Suppose w contains the pattern 231, i.e., there is a triple $i < j < k$ with $w_k < w_i < w_j$. Without loss of generality, we may assume that w_j is a peak. If neither w_i nor w_k are free, or if $s \notin \{w_i, w_k\}$, then we are done, as $H_s(w)$ leaves the relative positions of w_i, w_j , and w_j unchanged.

Now suppose $s = w_i$. Valley hopping would never allow w_i to move to the right of w_j (since $i < j$ and $w_i < w_j$), so $H_s(w) \notin S_n(231)$. Similarly, if $s = w_j$, $H_s(w) \notin S_n(231)$, since w_k would not be able to move to the left of w_j (since $k > j$ and $w_k < w_j$). See Figure 6. \square

We remark that the symmetric boolean decomposition of $(NC(n), \preceq)$ given here, as the image of $(S_n(231), \leq)$ under ϕ , is distinct from Simion

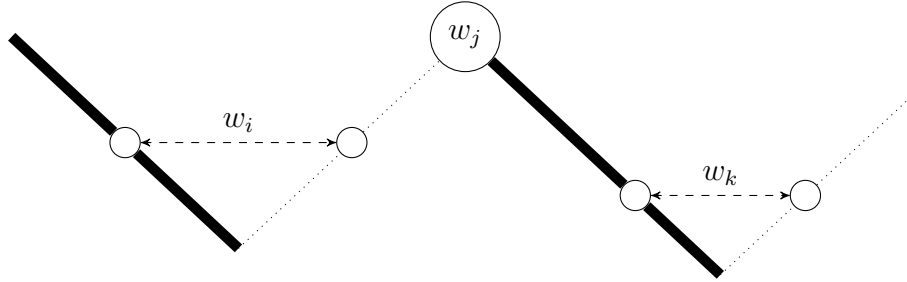


FIGURE 6. Valley hopping preserves the pattern 231.

and Ullman's decomposition. For example, in $NC(4)$, our decomposition describes the maximal boolean interval as:

$$P_{1234} = \{1|2|3|4, 21|3|4, 31|2|4, 41|2|3, 321|4, 421|3, 431|2, 4321\},$$

whereas Simion and Ullman's decomposition gives the maximal boolean interval as:

$$\{1|2|3|4, 21|3|4, 1|32|4, 1|2|43, 321|4, 21|43, 1|432, 4321\}.$$

6. FURTHER QUESTIONS

The natural next task is to exhibit a symmetric boolean decomposition for the shard intersection order (W, \leq) of any finite Coxeter group W . Ideally, this would restrict to a symmetric boolean decomposition of $(NC(W), \leq)$.

There is some hope for such a decomposition, as it is known that the W -Eulerian polynomial is γ -nonnegative for any W [18, Appendix]. One way to proceed in finding a symmetric boolean decomposition of the shard intersection order (W, \leq) is to take a purely combinatorial approach. Here, the first task is to gain combinatorial understanding of the shard intersection order for other classical reflection groups (i.e., B_n and D_n) just as Bancroft did for S_n . (For example, perhaps B_n shard intersections correspond to permutation pre-orders for S_{2n+1} that are invariant under 180-degree rotation.) Then with luck one would lift a combinatorial proof of γ -nonnegativity for the W -Eulerian polynomial as we have done here for $W = S_n$.

Of course a uniform proof would be best. We remark that at present there is no uniform proof of γ -nonnegativity for W -Eulerian polynomials.

Another reason to believe (W, \leq) might have a symmetric boolean decomposition is that, just as Simion and Ullman show for the classical (type A_{n-1}) lattice of noncrossing partitions, the lattice of noncrossing

partitions of type B_n has a symmetric boolean decomposition. This fact is proven by Hersh [9, Theorem 8], using the notion of an R^*S -labeling—a strong sort of chain-labeling together with a certain S_n -action on maximal chains. If one can show that the shard intersection order (W, \leq) has an R^*S labeling, then [9, Theorem 5] implies it has a symmetric boolean decomposition.

Another approach to Hersh’s result for $(NC(B_n), \leq)$ proceeds inductively as follows. In [15, Proposition 12], Reiner shows (as a prelude to proving the existence of a symmetric chain decomposition) that the type B_n noncrossing partition lattices are unions of products of smaller copies of noncrossing partition lattices with the appropriate centers of symmetry.¹ Thus, after checking small cases, it suffices to show that the product of two posets with symmetric boolean decompositions admits a symmetric boolean decomposition. This is true, as the following lemma shows.

Lemma 4. *Suppose (P, \leq_P) and (Q, \leq_Q) are posets with symmetric boolean decompositions. Then $(P \times Q, \leq_{P \times Q})$, with $(p, q) \leq_{P \times Q} (p', q')$ if and only if $p \leq_P p'$ and $q \leq_Q q'$, has a symmetric boolean decomposition.*

Proof. Suppose P has rank m and Q has rank n . Denote their respective decompositions by

$$\mathcal{P} = \{P_{i,j} : |P_{i,j}| = 2^{m-2j}\} \text{ and } \mathcal{Q} = \{Q_{k,l} : |Q_{k,l}| = 2^{n-2l}\}.$$

We wish to show $P \times Q$, which has rank $m+n$, has a symmetric boolean decomposition. We claim that the following is such a partition:

$$\mathcal{P}\mathcal{Q} = \{P_{i,j} \times Q_{k,l}\}.$$

Clearly $\mathcal{P}\mathcal{Q}$ is a partition of $P \times Q$.

Now, to check that Definition 2 is satisfied, we first see

$$|P_{i,j} \times Q_{k,l}| = |P_{i,j}| \times |Q_{k,l}| = 2^{m-2j} 2^{n-2l} = 2^{m+n-2(j+l)}.$$

Say p is the rank-minimal element of $P_{i,j}$, with $\text{rk}(p) = j$ in (P, \leq_P) , and q is the rank-minimal element $Q_{k,l}$ in (Q, \leq_Q) with $\text{rk}(q) = l$. Then (p, q) is the unique rank-minimal element of $P_{i,j} \times Q_{k,l}$, with $\text{rk}(p, q) = \text{rk}(p) + \text{rk}(q) = j + l$. Finally, since B_j is a subposet of $P_{i,j}$

¹Both Hersh and Reiner study other families of lattices of “signed” noncrossing partitions including a family they refer to as that of “type D_n ”. While these lattices admit symmetric boolean decompositions, this model has since come to be viewed as the wrong generalization from the point of view of root systems. In particular, it is different from the lattice $(NC(D_n), \leq)$ that lies inside the shard intersection order for D_n . See Athanasiadis and Reiner [1, Remark 4], where it is remarked that $NC(D_n)$ admits a symmetric chain decomposition.

and B_l is a subposet of $Q_{k,l}$, we have $B_{j+l} \cong B_j \times B_l$ is a subposet of $P_{i,j} \times Q_{k,l}$, as desired. \square

Corollary 1 ([17] and Theorem 8, [9]). *The noncrossing partition lattices $(NC(n), \leq)$ and $(NC(B_n), \leq)$ admit symmetric boolean decompositions.*

Recall that, for any finite Coxeter group W , the rank generating function of the W -noncrossing partition lattice is the h -polynomial of the corresponding W -associahedron [5, Theorem 5.9]. Thus, along with Observation 2, we have the following.

Corollary 2. *Let W be a Coxeter group of type A_{n-1} or B_n . Then the h -polynomial of the W -associahedron is γ -nonnegative.*

We remark that the coefficients of the h -polynomials of W -associahedra have nice formulas (these are W -Narayana numbers), and γ -nonnegativity for the classical types (including D_n) can be verified directly from these formulas. From [5, Figure 5.12] it is also straightforward to check that γ -nonnegativity holds for the W -associahedra of exceptional type.

We finish by observing that while we have provided two necessary conditions for a poset to have a symmetric boolean decomposition (Observation 1 and Observation 2), we have only one sufficient condition (Lemma 4). Hersh's work gives another sufficient condition: the existence an R^*S labeling for the poset [9, Theorem 5]. Just as others have given sufficient conditions for symmetric chain decompositions to exist [7, 8], so it may be of general interest to find other sufficient conditions for the existence symmetric boolean decompositions.

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